Quark confinement in the infinite-momentum frame

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We formulate the problem of quark confinement in the infinite-momentum frame. In this frame the dynamics is naturally described as a many-body problem: Quarks and gluons can be thought of as nonrelativistic particles moving in the two-dimensional transverse space with mechanical mass related to the longitudinal momentum P^+. In this language, a natural quark-confining mechanism is the condensation of gluons along a tube joining a separated quark and antiquark. This condensation is favored by two circumstances: (1) The fact that bare gluons are massless reduces the minimum energy for gluon pair production to zero, and (2) the octet color structure allows gluons to form into chains with long-range attractive nearest-neighbor interactions. We investigate the viability of this mechanism first in the limit $N_c \to \infty$, $N_c g^2$ fixed, where N_c is the number of colors in the theory. We analyze in detail a simplified version of the $N_c \to \infty$ dynamics which preserves the essential features of the full problem. This simplified model exhibits quark confinement and describes mesons as relativistic open strings. It also yields a relationship between the Regge slope α' and the scale μ_0, measured in deep-inelastic leptoproduction: $\mu_0^2 \approx 2/(\pi\sqrt{3}\alpha') \approx 0.4$ GeV^2, which is not too far from the experimental number 0.25 GeV^2. We discuss next the problem of finite N_c. We argue that the $1/N_c$ expansion is likely to have a vacuum instability which must be handled nonperturbatively before $1/N_c$ corrections can be calculated. We suggest that the true vacuum is a condensate of closed strings which have a finite density by virtue of repulsive interactions inherent in the four-gluon term in the Hamiltonian. A crude estimate of the condensate energy density yields an order-of-magnitude relation of the form $\epsilon \sim N_c^2/\alpha^2$. $\epsilon^{-1/4}$ should be a rough estimate of the thickness of the string.

I. INTRODUCTION

During the past few years, a number of promising schemes have been proposed to explain quark confinement as a dynamical consequence of a non-Abelian gauge theory such as quantum chromodynamics (QCD). While strong-coupling schemes based on lattice qauge theories[1] have provided valuable intuition about the confining phase, it is now clear that a real understanding of the phenomenon involves either an understanding of the continuum limit of lattice gauge theory or a reformulation of the problem in the continuum theory. As Wilson has emphasized,[2] the first program requires an implementation of renormalization-group ideas.[3] The second program has been initiated by Mandelstam[4] and by Callan, Dashen, and Gross.[5]

Mandelstam's scheme is motivated rather directly by the Meissner effect in superconductivity: He models the vacuum as a plasma of magnetic monopoles which confines color electric flux via the Meissner effect. The central dynamical problem in this scheme is to show that the monopole vacuum has lower energy than any state which allows color electric flux to spread out into all of space. So far he can show that the energy of his new vacuum is lower than that of the bare vacuum.

The approach of Callan, Dashen, and Gross is different in formulation from that of Mandelstam. They follow ideas of Polyakov[6] in which the problem is formulated in terms of a Euclidean path history approach to quantum mechanics. In this approach an approximate evaluation of the functional integral is attempted by saturating it with a dilute gas of "merons." The Wilson confinement criterion is satisfied in this evaluation, but the difficulty is to show that the dilute gas of merons does indeed dominate the functional integral. In this scheme and in Mandelstam's, the hope is that confinement will be described in a weak-coupling regime, with nonperturbative effects incorporated through quasisemiclassical calculations.

We finally mention the work of 't Hooft[7] which classifies the possible "phases" of a gauge theory as (1) normal with massless gluons and unconfined quarks, (2) Higgs mode in which color symmetry is spontaneously broken, or (3) a confined mode in which quarks are permanently trapped. Although this work does not introduce a new mechanism for confinement, it does provide a neat algebraic characterization of the different modes.

The various approaches we have just outlined are not in obvious conflict, although the details of the dynamical descriptions are superficially very different. Each approach provides a distinctive intuitive basis and calculational framework. A problem formulated in one framework may have an obvious solution which in another framework would be hard to find. There seems to be agreement that quark confinement involves a complex,

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subtle dynamics. The analogy to superconductivity has often been made. There also one has more than one framework to describe the phenomenon. The Bardeen-Cooper-Schrieffer (BCS) theory is of course the fundamental one, but the Ginzburg-Landau theory, although admittedly phenomenological, is indispensible for the understanding of such effects as vortices and phase boundaries.

In this article we describe yet another formulation of the problem of quark confinement—an infinite-momentum-frame description.[8] This approach has the virtue (drawback?) that the dynamics can be analyzed without having to find a complicated vacuum. Every dynamically interesting state carries finite energy momentum, so one deals directly with excitations above the vacuum.

The issue of confinement is simply the question of whether it is energetically favorable for gluons to condense along a tube joining a spatially separated quark and antiquark in such a way that there is finite energy per unit length along the tube. It seems likely that this will happen for a sufficiently large gauge group (i.e., a sufficiently large number of colors). The essential prerequisites for this condensation seem to be that the bare quanta are massless and that there is a net attraction between them. 't Hooft's $N_c \to \infty$ limit,[9] where N_c is the number of colors, seems to enforce these requirements and in this limit the tubes become strings of zero thickness, thus we make contact with dual dynamics.

Our calculational scheme does not introduce new parameters into the theory, and the mechanism for dimensional transmutation is explicit: this of course means that in the absence of quark masses we have no parameters at all, except for an overall scale. We illustrate this mechanism in a simplified model of the $N_c \to \infty$ dynamics and obtain for this model a relationship between the slope of Regge trajectories, α', and the scale μ_0 which characterizes the onset of asymptotic freedom,

$$\mu_0^2 \approx \frac{2}{\pi\sqrt{3\alpha'}} \approx 0.4 \text{ GeV}^2 ,$$

where μ_0 defines the scale in the running coupling constant, which in the $N_c \to \infty$ limit is

$$N_c \alpha_s(-q^2) = \frac{12\pi}{11 \ln(-q^2/\mu_0^2)} .$$

Fits to deep-inelastic data suggest that $\mu_0^2 \approx 0.25$ GeV^2, but we believe our simplified model is crude enough that this discrepancy is not serious.

We should perhaps emphasize that in this article we are describing an approach to the very complicated problem of performing calculations in a theory such as QCD. The actual calculations we have performed, such as the relation between μ_0^2 and α', are very preliminary in that they are made using over-simplified analog models of what we believe to be the essential physical mechanisms operating in QCD. The fact that these calculations yield the correct order-of-magnitude relationships between fundamental parameters is encouraging but certainly not conclusive. Our hope is that our calculational procedure can be refined to deal with the full problem at least in the $N_c \to \infty$ limit, without making such over-simplifications, but this is a problem for the future.

Our article is organized as follows. In Sec. II we review the (standard) formulation of QCD in the null plane gauge $A^+ = 0$. We find that the infrared divergence Mandelstam encounters in the timelike axial gauge is not present for the null-plane vacuum, but it does limit the dynamics of excited states. In Sec. III we describe the quark-confining mechanism as it appears in infinite-momentum frame. Section IV contains the bulk of the calculations we have so far carried out. We point out that 't Hooft's $N_c \to \infty$ limit singles out a subset of planar graphs with long-range attraction between gluons. We then reinterpret the sum over all $N_c \to \infty$ graphs as a many-body problem involving particles (gluons) ordered on a line and with only nearest-neighbor interactions. This problem is still formidable, and as a first step toward its solution we consider a model Hamiltonian which we believe retains the essential binding effects of the full problem. For this model Hamiltonian we can demonstrate the quark-confining mechanism and see how dimensional transmutation occurs. In Sec. V we give a brief summary of the results of the previous sections and discuss the problem of finite-N_c effects. We give a simple argument that our $N_c \to \infty$ closed string is unstable and indicate how this fact (if it does not destroy confinement) will give the string a thickness. The resulting picture of a fattened string is similar to long hadrons which occur in the MIT bag model. We suggested that the latter model may be a mean-field-theory description of hadronic states which occur in QCD.

II. QUANTUM CHROMODYNAMICS IN THE INFINITE-MOMENTUM FRAME

In this section we discuss the formulation of QCD in null-plane variables. Following standard treatments,[10] we take

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \tag{2.1}$$

to be our quantum evolution parameter, and we define

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)\,,$$
$$\vec{x} = (x^1, x^2)\,. \qquad (2.2)$$

The only gauge choice for which null-plane quantization is simple is

$$A^+ = -A_- = 0\,, \qquad (2.3)$$

which we henceforth adopt. The field strengths are given (as usual) by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]\,, \qquad (2.4)$$

where $A_\mu(x)$ is an anti-Hermitian $N_c \times N_c$ matrix. [The trace of A_μ is a free field in the absence of quarks, so we only need project it out of the quark-gluon vertex. For convenience we allow this free (singlet) field to be present in our formalism.] The QCD Hamiltonian is simply

$$P^- = -\tfrac{1}{2}\int d\vec{x}\,dx^- \{\mathrm{Tr}[F_{+-}{}^2] + \mathrm{Tr}[F_{12}{}^2]\} + P^-_{\text{quarks}}\,. \qquad (2.5)$$

In (2.5), F_{+-} is a dependent variable given by the formula

$$\frac{\partial F_{+-}}{\partial x^-} = -\frac{\partial}{\partial x^-}\vec{\nabla}\cdot\vec{A} - g[A_i, \partial_- A_i] + j^+_{\text{quarks}} \qquad (2.6)$$

and the quantum dynamics is completed by the commutation relations

$$[A_i^{\alpha\beta}(\vec{x}, x^-), -\partial_- A_j^{\beta'\alpha'}(\vec{0}, 0)] = \frac{i}{2}\delta(\vec{x})\delta(x^-)\delta^{\alpha\alpha'}\delta^{\beta\beta'}\delta_{ij}\,. \qquad (2.7)$$

It is very convenient to represent (2.7) by Fourier transformation in x^-:

$$A_i^{\alpha\beta}(\vec{x}, x^-) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dP^+}{\sqrt{2P^+}}[a_i^{\alpha\beta}(\vec{x}, P^+)e^{-ix^-P^+} - a_i^{\beta\alpha\dagger}(\vec{x}, P^+)e^{ix^-P^+}]\,, \qquad (2.8)$$

with

$$[a_i^{\alpha\beta}(\vec{x}, P^+), a_j^{\alpha'\beta'\dagger}(\vec{y}, Q^+)] = \delta_{ij}\delta^{\alpha\alpha'}\delta^{\beta\beta'}\delta(\vec{x}-\vec{y})\delta(P^+ - Q^+)\,. \qquad (2.9)$$

It is apparent from (2.6) and (2.8) that the $P^+ = 0$ point is in general singular. In particular, if the $P^+ = 0$ projection of the right-hand side of (2.6) is not zero, there will be an (infrared) infinite contribution to the energy (2.5). This is the light-cone version of the $k_3 = 0$ divergence in Mandelstam's formulation. In our case, however, the null-plane vacuum does not have this singularity; for

$$\int_{-\infty}^{\infty} dx^-(-g[A_i, \partial_- A_i] + j^+_{\text{quarks}})|0\rangle = 0\,. \qquad (2.10)$$

This is because (2.10) contains no term quadratic in its creation operators. In fact the state $|0\rangle$ is an exact eigenstate of P^-,

$$P^-|0\rangle = E_0|0\rangle\,, \qquad (2.11)$$

and we can measure all energies relative to E_0. Of course, it is still true that a general state $|\psi\rangle$ must be carefully chosen to avoid an infrared singularity in $\langle\psi|P^-|\psi\rangle$. $|\psi\rangle$ must at least be invariant under x^--independent gauge transformations which leave (2.3) invariant. An example of such a state would be

$$\bar{q}(\vec{x}, x^-)P\exp\left[ig\int_y^x d\xi\cdot\vec{A}(\xi)\right]q(\vec{y}, y^-)|0\rangle\,. \qquad (2.12)$$

Of course (2.12) is not an eigenstate of either P^- or P^+, but its structure indicates the need for gluons to accompany sources of color electricity.

It is incorrect to conclude from (2.12) that a physical string of gluons attaches the quark to the antiquark. In the case of quantum electrodynamics the path-ordered phase in (2.12) merely reflects the fact that soft photon bremsstrahlung accompanies any process involving the acceleration of charged particles. It is a question of dynamics whether gluons actually condense along a tube between the quark and antiquark with a finite energy per unit length. In the next section we explore this question in some detail.

III. A MECHANISM FOR QUARK CONFINEMENT

In the preceding section we have seen that the requirement that there be no infrared divergence in $\langle\psi|P^-|\psi\rangle$ led to the inclusion of the path-ordered phase in (2.12), but we stressed that this path had nothing to do with a physical string, as is clear from the example of QED. A proper understanding of quark confinement must explain at the same time why QED does not confine electrons and positions.

There is an amusing parallel to this subtlety in the distinction between ordinary conductors and superconductors, only the latter exhibiting the Meissner effect. The expression for the electronic current density is in either case

$$\vec{j}(\vec{x}) = \frac{i}{2}\psi^*(\vec{x})\vec{\nabla}\psi(\vec{x}) - \frac{e}{c}\vec{A}\psi^*\psi\,, \qquad (3.1)$$

and if the second term dominates this expression there would be a Meissner effect. When one calculates the current response in a metal to a weak external A field, there is also a contribution from the first term. For a free electron gas (which might model an ordinary conductor) the two terms cancel except for a weak Landau diamagnetism and there is no Meissner effect. In a superconductor the condensation of Cooper pairs causes a gap in the electron excitation spectrum. For sufficiently weak fields the contribution from the first term to the transverse current response is zero and there is a Meissner effect.

One may also describe this situation dynamically. If one builds up a magnetic field from zero, the instantaneous response of either a conductor or a superconductor is to exclude the magnetic field from its interior. The difference is that this situation is the lowest-energy state in a superconductor and the magnetic field can never penetrate into the interior. Whereas an ordinary conductor can lower its energy by allowing the magnetic field to penetrate in a characteristic time proportional to the conductivity.

In approaching the issue of quark confinement we must be cautious and consider carefully what may be wrong with a conventional interpretation of QCD describing quarks and massless gluons interacting weakly, *a la* conventional perturbation theory. We know we must include various instanton effects, but these are known not to yield confinement. It does not take much thought to realize there is a much more prosaic danger in perturbation theory. Gluons are interacting massless particles and in some channels these interactions are attractive. It could easily happen that these attractive forces bind gluons together into condensed lumps of matter, in which case perturbation theory would break down. Indeed, one possible interpretation of the pole in the leading-logarithm approximation to the renormalization invariant charge,

$$g^2(t)=\frac{g^2(-\mu^2)}{1+bg^2(-\mu^2)\ln(-t/\mu^2)}, \tag{3.2}$$

is that such binding is taking place. The famous asymptotically free sign is indeed associated with the predominantly attractive self-gluon interactions. This argument for a breakdown of perturbation theory is of course not rigorous since the approximations leading to (3.2) are only valid for $|t|$ large, and one cannot rule out the possibility that the exact $g(t)$ is not singular for finite t. Nonetheless gluon condensation is a possibility, and in this article we are exploring the consequences of such an effect for quark confinement.

The scenario we envisage is pictured in Fig. 1. The gluon field is a matrix $A^{\alpha\beta}$, α being a 3 index and β being a $\overline{3}$ index. We therefore picture a gluon as sort of a $3\,\overline{3}$ dipole: There are attractive forces between the 3 index of one gluon and the $\overline{3}$ index of another gluon. It is possible that the energetics is such that a chain of gluons will condense along a line joining a spatially separated quark and antiquark. If this chain has a finite en-

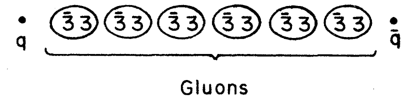

FIG. 1. Possible mechanism for quark confinement. Gluons condense along a line, locally canceling the color-triplet charge of the quark and antiquark.

ergy per unit length, then quark confinement would be a reality. Of course this scenario is very similar to that in strong-coupling lattice gauge theory where it is a reality because the potential energy is forced to dominate.

In the next section we develop plausibility arguments that such a condensation could easily occur without making a strong coupling assumption and without introducing a lattice. The infinite-momentum-frame dynamics discussed in Sec. II will be a great aid in developing this intuition. Our scheme for quark confinement will amount to an ansatz within which calculations can be performed in great enough detail so that nontrivial self-consistency checks can be made. All quantities are calculable within the ansatz, and in particular dimensional transmutation is explicit.

IV. A DETAILED CALCULATIONAL FRAMEWORK FOR THE DESCRIPTION OF QUARK CONFINEMENT

We should perhaps begin this discussion by reminding the reader of some of the novelties of infinite-momentum-frame calculations.[11] The fact that the null-plane vacuum is passive is, of course, the most striking feature of this approach. We have already seen that this feature avoids an immediate crisis with an infrared divergence in the $F_{+-}{}^2$ term of the Hamiltonian. The essential feature here is that the Fourier decomposition (2.8) has a natural separation of positive P^+ and negative P^+ components. The unique state with $P^+=0$ is the bare vacuum $|0\rangle$ which satisfies

$$a(\vec{x},P^+)|0\rangle=0\,. \tag{4.1}$$

Every other state of the theory is of the form

$$\text{polynomial}(a^\dagger(\vec{x},P^+))|0\rangle$$

and has total $P^+\geq 0$. In order to handle the $P^+=0$ point, we shall establish a temporary infrared cut off by discretizing P^+ (Ref. 12):

$$P^+=lb,\quad l=1,2,\ldots, \tag{4.2}$$

with the understanding that $b\to 0$ at the end of the calculation. Let us spell this $b\to 0$ limit out a little more carefully. Because P^+ is a conserved quantum number, we can decide to work at fixed total P^+. Because of the cutoff (4.2) P^+ will be a large integer multiple of b:

$$P^+_{\text{total}} = Mb\,. \tag{4.3}$$

The dynamics will never carry us out of this P^+ sector and we uniquely specify the $b \to 0$ limit by holding Mb fixed. Note that for fixed P^+ and b, there are a maximum number of quanta which can be present in the state, namely M.

The Fourier transform (2.8) takes the form

$$A_i^{\alpha\beta}(\vec{x}, x^-) = \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{\infty} \frac{1}{\sqrt{2l}} \left[a_{il}^{\alpha\beta}(\vec{x}) e^{-ix^- lb} - a_{il}^{\beta\alpha\dagger}(\vec{x}) e^{ix^- lb}\right] \tag{4.4}$$

and (2.9) the form

$$[a_{il}^{\alpha\beta}(\vec{x}), a_{jl'}^{\alpha'\beta'\dagger}(\vec{y})] = \delta_{ij}\delta^{\alpha\alpha'}\delta^{\beta\beta'}\delta_{ll'}\delta(\vec{x}-\vec{y})\,. \tag{4.5}$$

When these and analogous expressions for quark fields are plugged into the Hamiltonian, the reader will recognize that the problem has been converted into a nonrelativistic many-body problem in two space dimensions, with, however, an infinite number of species of particles labeled by l, the nonrelativistic mass of the particle. It is an unusual many-body problem in that mass can be transferred between particles and there are also processes in which a particle can fission into several particles, or several particles can fuse into a single one.

We can now begin to formulate the problem of gluon condensation in a very concrete way. The issue is whether a state containing many gluons, by virtue of attractions, has a lower P^- than one containing only a few, and whether these many gluons form into clumps. If the gluons had mass, there would be terms such as

$$\frac{m^2}{2P^+}$$

in the Hamiltonian which would suppress low P^+ and would prevent an infinite condensation, so one prerequisite is that the quanta be massless. Another prerequisite is that there is a net attraction between gluons.

We know that the gluon force has both attractive and repulsive components depending on the quantum numbers. If we tentatively focus on the long-range gluon-gluon force, it is given in perturbation theory by the diagrams in Fig. 2. The arrows indicate the flow of color. The reader will recognize that the diagram in Fig. 2(a) is attractive in the color channel where the $3\,\bar{3}$ indices connected by the gluon form a color singlet and the diagram in Fig. 2(b) is predominantly repulsive at long distances. Now both diagrams are the same strength; but when they are iterated to develop an effective potential, they have very different behaviors. The iteration of Fig. 2(a) enhances the attractive singlet channels by factors of N_c over the repulsive channels and over the iteration of Fig. 2(b). Thus if $N_c \gg 1$ the attractive interaction is enhanced over the repulsive interaction.

Now consider the many-body potential described by Fig. 3. Its iteration will similarly enhance the attractive singlet channels over the iteraction of any of the potentials with crossed color lines. We therefore conclude that the graphs with planar color flow contain mainly attractive long-range gluon-gluon forces and it is precisely these graphs which are enhanced if $N_c \gg 1$. Thus we adopt a new attitude toward 't Hooft's $N_c \to \infty$ limit with $N_c g^2$ fixed.[9] It is a way of selecting a gauge-invariant subset of graphs which contain the important long-range attractive forces between gluons. From this point of view it seems eminently sensible to assess the possibility of quark confinement in this limit first as a necessary (but not sufficient) condition for confinement at finite N_c. If we have the right confinement mechanism, this limit should optimize our chances. We shall have a little more to say about finite-N_c effects in the succeeding section.

Even after taking the $N_c \to \infty$ limit we still have a formidable problem to solve, i.e., we must sum an infinite sum of planar diagrams, and we have not yet succeeded in doing this. However, we can gain considerable insight into the solution of the problem by studying a simplified model of the $N_c \to \infty$ dynamics which shares what we believe are the essential features of the $N_c \to \infty$ limit. Let us first describe these features. Our infinite-momentum-frame description has converted QCD into a many-body problem involving a system of mutually interacting quarks and gluons. The only interactions which survive in the $N_c \to \infty$ limit are described by planar diagrams with a minimal num-

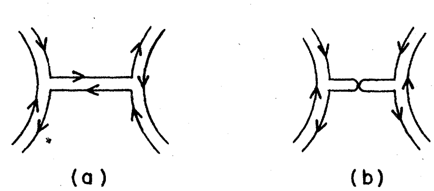

FIG. 2. Feynman diagrams containing the long-range gluon-gluon force. (a) is attractive and (b) is repulsive at long distances.

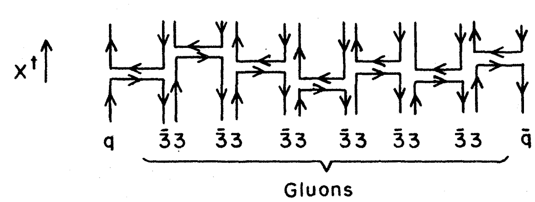

FIG. 3. Diagram describing many-body force responsible for gluon condensation.

ber of quark loops. By taking an equal-x^+ cut of a generic planar diagram with, say, one quark loop, we see that the many-body problem is in fact a chain problem: The ends of the chain are a quark and antiquark, and the constituents of the chain are gluons. Planarity means that only nearest neighbors interact and we know from the preceding paragraph that these nearest-neighbor interactions are attractive and spin independent at least at long distances. The second important feature we should include is the fact that gauge invariance assures that the perturbative interactions do not give the gluons a rest mass, so the kinetic energy required to add gluons to the system is minimal. The third and last feature of QCD we will include is the scale invariance of the interactions.

We now present a simple many-body chain problem which shares these three features of the $N_c \to \infty$ limit:

(1) attractive nearest-neighbor interactions,
(2) massless constituents,
(3) scale invariance.

Consider a system of M' particles, each carrying an amount b' of P^+ and described by the Hamiltonian P^-:

$$P^- = -\frac{1}{2b'}\sum_{i=1}^{M'}\frac{\partial^2}{\partial \vec{x}_i^{\,2}} + \sum_{i=1}^{M'-1} \mathcal{V}(\vec{x}_{i+1}-\vec{x}_i)\,, \qquad (4.6a)$$

where the wave function depends on the transverse coordinates $\vec{x}_i$. This model suppresses the role of gluon spin, P^+ exchange and gluon creation and annihilation, but does exhibit features (1) and (2) if $\mathcal{V}(\vec{x})$ is attractive. The simplest choice for $\mathcal{V}(\vec{x})$ which satisfies (3) is

$$\mathcal{V}(\vec{x}) = -\frac{\lambda_0}{b'}\delta(\vec{x})\,, \qquad (4.6b)$$

and we shall use this choice for numerical evaluations. The fact that gluon creation and annihilation has been suppressed means that $M' = P^+/b'$ is a free parameter: States with different numbers of gluons do not mix in our model. If weak mixing were turned on, the system would choose M' to minimize the energy. We therefore roughly take this into account by treating M' as a variational parameter.

Before we discuss (4.6), we should remark on the crudeness of its representation of the full problem. We have replaced a complicated gluon-gluon potential, which is in general long range and momentum dependent, by a short-range contact interaction. We might hope that screening will make this replacement a lot less atrocious than it seems. We shall see that the final result is largely independent of the details of the potential. A more severe limitation of (4.6) is that we have assumed each particle has the same P^+, b' and have forbidden particle destruction, creation, and P^+ transfer between particles. We shall see that $M' \to \infty$ lowers the energy and so $b' \to 0$ and no constituent will carry a finite amount of P^+. Our representation thus forbids the occurrence of valence partons and thus an important phenomenological feature will be missing. We should get a decent description of wee partons. We have hopes that this limitation can be removed by later refined calculations. Finally we have neglected spin-spin and spin-orbit forces which are important for the hyperfine structure.

Let us first establish that our model many-body system in fact binds together. That is we want to establish that it costs energy to separate the system into two clumps. We first take out the center of mass by defining

$$\vec{X} = \frac{1}{M'}\sum_{i=1}^{M'}\vec{x}_i\,,$$
$$\vec{y}_i = \vec{x}_{i+1}-\vec{x}_i\,, \quad i=1,2,\ldots,M'-1\,, \qquad (4.7)$$

so that (4.6) becomes

$$\begin{aligned} P^- &= -\frac{1}{2P^+}\frac{\partial^2}{\partial \vec{X}^2} - \frac{1}{b'}\sum_{i=1}^{M'-1}\frac{\partial^2}{\partial \vec{y}_i^{\,2}} + \frac{1}{b'}\sum_{i=1}^{M'-2}\frac{\partial^2}{\partial \vec{y}_i\cdot\partial \vec{y}_{i+1}} - \frac{\lambda_0}{b'}\sum_{i=1}^{M'-1}\delta(\vec{y}_i) \\ &= \frac{\vec{P}^2}{2P^+} + \frac{1}{b'}\left\{\sum_{i=1}^{M'-1}\left[-\frac{\partial^2}{\partial \vec{y}_i^{\,2}} - \lambda_0\delta(\vec{y}_i)\right] + \sum_{i=1}^{M'-2}\frac{\partial^2}{\partial \vec{y}_i\cdot\partial \vec{y}_{i+1}}\right\} \\ &\equiv \frac{\vec{P}^2}{2P^+} + \frac{E_{M'}}{b'}\,, \end{aligned} \qquad (4.8)$$

where

$$\left\{\sum_{i=1}^{M'-1}\left[-\frac{\partial^2}{\partial \vec{y}_i^{\,2}} - \lambda_0\delta(\vec{y}_i)\right] + \sum_{i=1}^{M'-2}\frac{\partial^2}{\partial \vec{y}_i\cdot\partial \vec{y}_{i+1}}\right\}\psi(\vec{y}_1,\ldots,\vec{y}_{M'-1}) = E_{M'}\psi\,.$$

Let $E^G_{M'}$ be the ground-state energy of the M' body system. Using the variational principle one can establish rigorously the inequality

$$E^G_{M+N} \leq E^G_M + E^G_N + E^G_2 \qquad (4.9)$$

by taking a trial

$$\psi_{M+N}=\psi_M^G\psi_N^G\psi_2^G(\vec{y})\,. \tag{4.10}$$

If $E_2^G<0$, i.e., if the two-body system forms a bound state, then (4.9) establishes that the ground state is absolutely stable, with an ionization P^- of at least $(1/b)|E_2^G|$.

So consider the problem

$$\left[-\frac{\partial^2}{\partial\vec{y}^2}-\lambda_0\delta(\vec{y})\right]\psi(\vec{y})=E_2\psi(\vec{y})\,. \tag{4.11}$$

In momentum space this equation is

$$\phi(\vec{p})=\frac{\lambda_0}{2\pi}\frac{\psi(\vec{0})}{\vec{p}^2-E_2}\,,$$

from which follows the eigenvalue condition

$$2\pi=\frac{\lambda_0}{2\pi}\int\frac{d\vec{p}}{\vec{p}^2-E_2}=\frac{\lambda_0}{2}\ln\left(\frac{\Lambda^2}{-E_2}\right).$$

Thus the ground-state energy is

$$E_2^G=-\Lambda^2e^{-4\pi/\lambda_0}\,, \tag{4.12}$$

a relation characteristic of dimensional transmutation. Note that the binding energy is in the infrared, i.e., much smaller than Λ^2, provided $\lambda_0\ll1$, characteristic of asymptotically free theories.

Applying (4.9) repeatedly leads to the inequality

$$E_M^G\leq(M'-1)E_2^G<0\,, \tag{4.13}$$

and this estimate for the ground-state energy corresponds to the trial

$$\psi_M^G=\prod_{i=1}^{M'-1}\psi_2^G(\vec{y}_i)\,,$$

which is a Hartree-Fock–type approximation. If b' is treated as a variational parameter, (4.13) verifies that the lowest energy state corresponds to $b'\to0$, so in this limit the ionization $P^-=|E_2^G|/b'\to\infty$.

Finally, we must seek the low-lying excitations of the system which are to be identified, e.g., with Regge recurrences. Goldstone[13] has applied a sort of random phase approximation (RPA) to this system and has shown within this approximation that there are indeed finite P^- excitations as $b'\to0$. The following discussion is due to him.

We must analyze the Hamiltonian

$$H\equiv b'P^-_{\text{internal}}=\sum_{r=1}^{M'-1}(\vec{p}_r^{\,2}+\mathcal{V}(\vec{y}_r))-\sum_{r=1}^{M'-2}\vec{p}_r\cdot\vec{p}_{r+1}\,. \tag{4.14}$$

The analysis is simplified if we add to H a weak time-dependent forcing term

$$\sum_r\vec{p}_r\cdot\vec{f}_r(t)$$

and calculate the linear response

$$\langle p_{ri}(t)\rangle_f\equiv\int G_{rs}^{ij}(t-t')f_{sj}(t')dt'\,. \tag{4.15}$$

The Fourier transform of $G_{rs}^{ij}(t)$ has poles at the eigenvalues of H in (4.14).

The RPA is to replace $H+\sum_r\vec{p}_r\cdot\vec{f}_r$ by

$$H_{\text{RPA}}(\vec{f}_r)\equiv\sum_r[\vec{p}_r^{\,2}+\mathcal{V}(\vec{y}_r)]+\sum_r\vec{p}_r\cdot(\vec{f}_r-\langle\vec{p}_{r+1}\rangle-\langle\vec{p}_{r-1}\rangle)\,. \tag{4.16}$$

One can now calculate $\langle p_{ri}(t)\rangle$ in terms of the one-body problem

$$h=\vec{p}^2+\mathcal{V}(\vec{y})+\vec{p}\cdot\vec{f}(t)$$

for which the response to $\vec{f}$ is defined by

$$\langle p_i(t)\rangle_{\vec{f}}\equiv\int dt'\hat{G}^{ij}(t-t')f_j(t')\,.$$

Thus

$$\langle p_{ri}(t)\rangle_{\vec{f}_r}=\int dt'\hat{G}^{ij}(t-t')\left\{f_{rj}(t')-\int dt''[G_{r+1,s}^{jk}(t'+t'')+G_{r-1,s}^{jk}(t'-t'')]f_{sk}(t'')\right\},$$

i.e.,

$$G_{rs}^{ij}(t-t')=\hat{G}^{ij}(t-t')\delta_{rs}-\int dt''\hat{G}^{ik}(t-t'')[G_{r+1,s}^{kj}(t''-t')+G_{r-1,s}^{kj}(t''-t')]\,.$$

Now, define the Fourier transforms:

$$\hat{G}^{ij}(t)=\frac{1}{2\pi}\int d\omega\,\hat{\mathcal{G}}^{ij}(\omega)e^{-i\omega t}\,,\qquad G_{rs}^{ij}(t)=\frac{1}{2\pi}\int d\omega\,\mathcal{G}_{rs}^{ij}(\omega)e^{-i\omega t}\,, \tag{4.17}$$

so the above recursion relation becomes

$$\mathcal{G}_{rs}^{ij}(\omega)=\hat{\mathcal{G}}^{ik}(\omega)[\delta^{kj}\delta_{rs}-\mathcal{G}_{r+1,s}^{kj}(\omega)-\mathcal{G}_{r-1,s}^{kj}(\omega)]\,.$$

This recursion relation may be solved by normal modes:

$$\mathcal{G}_{rs}^{ij}(\omega)=\frac{2}{M'}\sum_{n=1}^{M'-1}\sin\frac{r\pi n}{M'}\sin\frac{s\pi n}{M'}\mathcal{G}_n^{ij}(\omega)$$

so that

$$\left(\delta^{ik}+2\,\hat{\mathcal{G}}^{ik}(\omega)\cos\frac{n\pi}{M'}\right)\mathcal{G}_n^{kj}(\omega)=\hat{\mathcal{G}}^{ij}(\omega)\,. \tag{4.18}$$

An explicit formula for $\hat{G}^{ij}$ is

$$\hat{G}^{ij}(t-t')=i\theta(t-t')\langle\langle p_i(t)p_j(t')-p_j(t')p_i(t)\rangle\rangle_{\vec{f}=0}$$
$$=i\tfrac{1}{2}\delta_{ij}\theta(t-t')\langle\langle\vec{p}(t)\cdot\vec{p}(t')-\vec{p}(t')\cdot\vec{p}(t)\rangle\rangle \tag{4.19}$$
$$\hat{\mathcal{G}}^{ij}(\omega)=\delta_{ij}\sum_n\frac{\langle 0|\vec{p}|n\rangle\cdot\langle n|\vec{p}|0\rangle(E_n-E_0)}{\omega^2-(E_n-E_0)^2}\,.$$

The excitation energies of the many-body system are given by the locations of the poles in $\mathcal{G}_n^{kj}(\omega)$ which from (4.18) are the locations of the zeros of

$$\delta^{ik}+2\,\hat{\mathcal{G}}^{ik}(\omega)\cos\frac{n\pi}{M'} \tag{4.20}$$

and low-lying excitations will occur in the limit of large M' if $\mathcal{G}^{ik}(0)=-\frac{1}{2}\delta^{ik}$.

The $\omega\to 0$ limit of (4.19) can be obtained using the relation $\langle 0|\vec{p}|n\rangle=\frac{1}{2}(E_n-E_0)\langle 0|\vec{x}|n\rangle$:

$$\hat{\mathcal{G}}^{ij}(0)=-\tfrac{1}{4}\delta^{ij}\sum_n\langle 0|\vec{x}|n\rangle\langle n|\vec{x}|0\rangle(E_n-E_0)$$
$$=\tfrac{1}{8}\delta^{ij}\sum_i\langle 0|[x_i,[x_i,H]]|0\rangle=-\tfrac{1}{2}\delta_{ij}\,, \tag{4.21}$$

as required. Thus for a general potential the low-lying excitations are, in the limit of large M', given by

$$1-\cos\frac{n\pi}{M'}-2\omega^2\sum_{m>0}\frac{\langle 0|\vec{p}|m\rangle\langle m|\vec{p}|0\rangle}{(E_m-E_0)^3}=0$$

or

$$\omega_n\left(\sum_{m>0}\frac{\langle 0|\vec{p}|m\rangle\langle m|\vec{p}|0\rangle}{(E_m-E_0)^3}\right)^{1/2}=\frac{n\pi}{2M}\,,$$

so that

$$(P_n^- - P_0^-)\left(\sum_{m>0}\frac{\langle 0|\vec{p}|m\rangle\langle m|\vec{p}|0\rangle}{(E_m-E_0)^3}\right)^{1/2}=\frac{n\pi}{2P^+}\,. \tag{4.22}$$

The P^+ dependence of (4.22) is in accord with the requirements of Lorentz invariance, a nontrivial consistency check. From (4.22) we infer that the slope of Regge trajectories is given by

$$\alpha'=\frac{1}{\pi}\left(\sum_{m>0}\frac{\vec{p}_{0m}\cdot\vec{p}_{m0}}{(E_m-E_0)^3}\right)^{1/2}\,. \tag{4.23}$$

For our δ potential the formula for α' is

$$\alpha'=\frac{1}{\pi\sqrt{12}\,|E_2|}\,. \tag{4.24}$$

$(1/b)|E_2|$ is the binding energy of the two-body potential, or $-4|E_2|$ is invariant (mass)^2. Equation (4.23) is remarkable in that it relates the hadronic scale associated with a collective phenomenon, Regge behavior, to a parameter $|E_2|$ associated with the force between elementary quanta. We might hope that a refined version of our calculational framework would yield a calculation of α' in terms of deep-inelastic data.

Indeed, even at this point we might crudely associate $4|E_2|\equiv\mu_0^2$ with a typical energy scale in deep-inelastic phenomena. Taking $\alpha'=1\ \text{GeV}^{-2}$, in (4.24) yields

$$\mu_0^2\approx 0.4\ \text{GeV}^2\,. \tag{4.25}$$

This number determines the onset of asymptotic freedom. For example, one can use the model Hamiltonian (4.8) to calculate the running coupling constant governing the scattering of two elementary quanta:

$$\frac{\lambda(-q^2)}{4\pi}=\frac{1}{\ln(-q^2/\mu_0^2)}\,.$$

Fits to deep-inelastic data suggest a value for μ_0 around 500 MeV or for μ_0^2 of 0.25 GeV^2. Our number (4.25) is nearly a factor of two larger, but we feel this is not bad agreement considering the roughness of the calculation. This preliminary calculation does indicate that our model may have enough predictive power to be tested in both the scaling and Regge regions.

V. DISCUSSION AND CONCLUDING REMARKS

In the preceding section we pushed through to the end a calculation of the spectrum of a system of q, $\bar{q}$, and gluons, making approximations and simplifications left and right, with the main object of illustrating our quark-confining mechanism and seeing how dimensional transmutation arises. In this section we shall review these approximations and discuss to what extent they are essential.

We believe that the most drastic approximation was the $N_c\to\infty$ limit, and this more because of the implied interchange of limits than that $N_c=3$ is not large. (Indeed, 30% accuracy for a zero-parameter calculation should be regarded as astounding success). But this approximation is also mathematically well defined and one can imagine systematically calculating $1/N_c$ corrections. We shall devote the bulk of this section to this subject, after we have discussed our handling of the $N_c\to\infty$ calculation themselves. At first sight, our replacement of the full sum of $N_c\to\infty$ graphs by the many-body problem Eq. (4.6) seems very crude indeed. But in retrospect, one realizes that the final results are insensitive to many of the details of this approximation. For example, the only feature of the two-body potential which is essential to our results is the existence of a bound state, all other aspects of the potential being lumped into the numerical value of α' [Eq. (4.23)]. Thus the only essential limitation of (4.6) is the assumption of a uniform P^+ distribution; in effect we are including only an infinitesimal bit of P^+ phase space in the problem, the assumption being that the dynamics enhances this part of phase space over all the rest. The fact that taking $b' \to 0$ lowers the energy is a reflection of this dynamical effect: The mutual attraction favors the presence of many gluons which in turn must share a finite amount of P^+ among themselves, thus limiting the available P^+ phase space per gluon.

If the quarks are massive, we would certainly expect the P^+ distribution of the gluons near them to be nonuniform and not accurately described by our model Hamiltonian, and in general the absence of finite-momentum partons in the ground-state wave function is a defect of (4.6). We believe (4.6) does give an accurate picture of the build up of the confining force via wee-parton (gluon) condensation. We should reiterate that our calculation is only a first crude indication of what the sum of the $N_c \to \infty$ graphs might look like. We believe that the full problem should in fact be soluable at least numerically. The limit has converted the problem effectively to a $(1+1)$-dimensional field theory, the fields being $x_i(x^+)$, $P_i^+(x^+)$ and perhaps gluon spin variables $S_i(x^+)$, and it is just this type of problem (two-dimensional statistical mechanics) that many-body physicists are making great progress with.

We would like to compare these results on $N_c \to \infty$ to previous work on strong-coupling fishnet diagrams on x^+, P^+ lattices.[12,14] The results of Sec. IV indicate that the strong-coupling assumption is not necessary, nor is the ultraviolet cutoff (i.e., we now have continuous x^+). The infrared cutoff remains in (4.6). We have suppressed gluon spin dynamics but it seems very likely that the results of Brower, Giles, and Thorn[14] may describe these effects in the wee-parton approximation. This approximation seems to naturally suppress spin-orbit forces and it will clearly be worthwhile to get beyond wee partons.

But suppose now we were given the exact solution to the $N_c \to \infty$ problem, and it had all the qualitative features we expect, in particular quark confinement. What, if anything, does this teach us about the problem of interest, namely $N_c = 3$, or even about the problem of large but finite N_c? One possibility[15] is that one can define an orderly expansion in $1/N_c$ which would resemble the dual-loop expansion. That is, the $N_c \to \infty$ model does not change qualitatively for large but finite N_c, except that hadronic resonances acquire finite lifetimes ($\Gamma \sim 1/N_c$) because $q\bar{q}$ pair production is allowed to first order in $1/N_c$.

We should interpolate here, that the $N_c \to \infty$ limit, among other simplifications, simplified the criterion for quark confinement. Because $q\bar{q}$ production is $O(1/N_c)$, confinement is simply the statement that it takes an infinite energy to separate the q and $\bar{q}$ an infinite distance. When we address the issue of finite $1/N_c$ effects on quark confinement we shall relegate quarks to be external sources so the confinement criterion is again simple.

If the $1/N_c$ expansion is to be well defined, it must be true that the $N_c \to \infty$ limit gives a zero-width model with no tachyons. We now give an argument that indicates that tachyons in the closed-string sector are very hard to avoid. The problem is that the $N_c \to \infty$ limit allows only nearest-neighbor bonds. Compare a closed string with $P^+ = P_1^+ + P_2^+$ with two closed strings with $P^+ = P_1^+, P_2^+$, respectively (see Fig. 4). If we think of the strings as consisting of particles held together by bonds, one can pass from (a) to (b) conserving the number of bonds and it seems reasonable that this should not cost potential energy. On the other hand, the kinetic energy in (b) can be lowered by taking the two strings far apart, so we expect

$$\min(P_1^- + P_2^-) < P_{(1+2)}^{-G}\ ,$$

and this is the tachyon instability: A large closed string is unstable for evaporation into a large number of closed stringlets. This argument is actually rigorous if the P^+ distribution of the string constituents is assumed uniform. Although it is conceivable that clumping effects could get around the argument, we feel it is sufficiently compelling to force us to seriously consider the possibility that the $1/N_c$ expansion breaks down due to vacuum instability.

The argument of the previous paragraph strongly suggests that one cannot perform a simple $1/N_c$ expansion. But, just as with field theories with spontaneous symmetry breaking, this does not imply that large N_c calculations cannot be performed, but only that the instability be handled nonperturbatively before proceeding with the expansion. We can identify two approaches to the problem.

The first is to go back to the full problem and try to identify these nonperturbative effects as N_c gets large but not infinite. For large N_c, we should really only say that the planar graphs are being

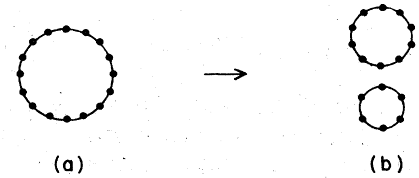

FIG. 4. A closed string may be unstable to decay into two smaller closed strings.

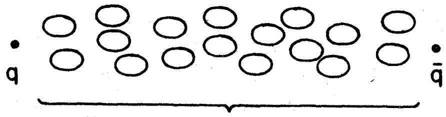

FIG. 5. For finite N_c, gluons are not completely ordered on a line, and the string acquires thickness.

enhanced, but that there are many more nonplanar graphs which describe non-nearest-neighbor interactions which are both attractive and repulsive. In these nonplanar graphs, the long-range attractive and repulsive components of the forces are roughly of the same strength and will tend to cancel each other. Thus for very large N_c there would be no rigid ordering of particles but only a tendency to order because the ordered graphs are enhanced by factors of N_c as well as containing consistently attractive forces. So for finite N_c we have a picture more like Fig. 5, with the string having a thickness which goes to zero as $N_c \to \infty$. It is obvious that to finite order in $1/N_c$, the string can never acquire a thickness over its whole length. So the problem of instability might well be resolved by keeping the finite-thickness effects as one goes to $N_c \to \infty$. This procedure would avoid the above argument for instability. If this is the resolution we can then identify the regime for validity of the $N_c \to \infty$ calculations: This would be for processes in which the typical distance between the quarks is much larger than the thickness induced by $1/N_c$ effects. Thus they should apply to long hadron calculations as well as high-energy peripheral interactions, precisely the domain where duality ideas are most successful.

In order to estimate $1/N_c$ effects we shall adopt a different approach. This is to interpret the naive $1/N_c$ expansion as a weak-coupling expansion around a wrong vacuum. With respect to this wrong vacuum the $N_c \to \infty$ spectrum is noninteracting open and closed strings with the ground-state closed string being a scalar tachyon. As $1/N_c$ is turned on, the bare strings start emitting closed strings into the vacuum. It is clear that the condition for stability is that repulsive interactions between the closed-string tachyons determine a limiting density of strings in the vacuum. It is the presence of these repelling closed strings in the vacuum that stabilizes the string configurations.

We now crudely estimate these effects by representing the closed string tachyon as an effective scalar field $\phi(x)$, with an effective Lagrangian density,

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 . \qquad (5.1)$$

We may identify μ^2 with $1/\alpha'$, the scale of excitations of the bare strings. To estimate λ we must

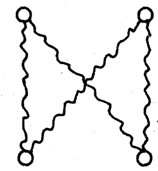

FIG. 6. A diagram giving rise to a repulsive closed-string–closed-string contact interaction. This one represents the scattering of a pair of two-gluon systems. For topological reasons the strength of the diagram is $(N_c g^2)/N_c^2$.

identify an effective contact interaction between closed strings. In spite of the fact that $N_c g^2$ is not a small parameter, we boldly identify this interaction as the one we obtain in lowest-order perturbation theory in $N_c g^2$. Thus

$$\lambda = c\,\frac{8\pi\alpha_s N_c}{N_c^2} ,$$

where c is a pure number of order (1). For example the graph indicated in Fig. 6 yields a repulsive contact interaction of this form with $c = 1$ (see the Appendix).

The physical quantity we would like to extract from this rough physics is the energy density ϵ, associated with the condensate. This is

$$\epsilon = \frac{3\mu^4}{2\lambda} \approx \frac{9}{16\pi\alpha_s\alpha'^2 c} . \qquad (5.2)$$

We should probably choose g^2 appropriate to the scale set by α' for which $\alpha_s \approx 1$. This formula should only be taken as a rough indication of the relation between parameters in our scheme, in particular the value of c depends on very complicated dynamics.

Figures 1 and 5 remind us of a kind of color electric Meissner effect. The gluon matter between the q and $\bar{q}$ possesses a kind of color electric dipole moment density which effects a local cancellation of color charge down the tube connecting the quark and antiquark. The analogous situation for superconductivity would be to place a north and a south magnetic monopole inside a vast superconductor. Because of the Meissner effect, the response of the superconductor is to produce solenoidal currents surrounding a tube joining the monopoles flowing in just such a way as to cancel the magnetic field in the bulk of the superconductor and necessarily collimating the magnetic flux into the tube (see Fig. 7). Thus the current vortex plays the same role in superconductivity as the gluon matter does in our model for quark confinement. In fact we could formally replace the gluonic tube in Fig. 5 with a solenoidal current distribution of magnetic monopoles and obtain the same qualitative physics (as long as we do not probe the microscopic details). This "dualized" model resembles very closely the MIT

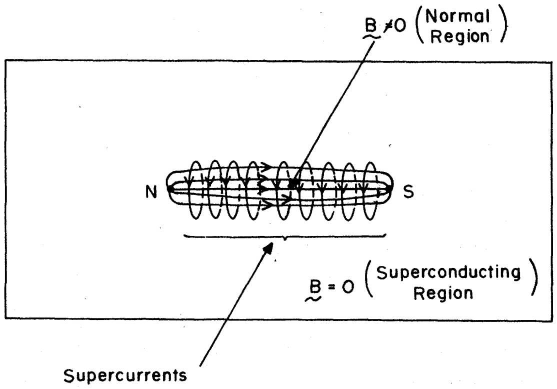

FIG. 7. Magnetic monopole confinement in a superconductor. The solenoidal pattern of supercurrents sets up an effective dipole distribution which, so to speak, locally cancels the magnetic charge of the north and south monopoles. This mechanism is analogous to Fig. 1.

bag model, even explaining the color electric Meissner effect. In such an analogy, we could roughly identify ϵ (5.2) with the bag constant

$$B \sim \epsilon .$$

Using this analogy we can interpret $\epsilon^{1/4}$ as a confining pressure which determines the thickness of the tube to be roughly $\epsilon^{-1/4}$. Putting $\alpha_s \approx 1$, and using the relation between α' and $1/\mu_0^2$ from our model calculation [Eq. (4.25)] yields

$$B \sim \frac{27\pi}{64c} \mu_0^4$$

or

$$B^{1/4} \sim \frac{1.1}{c^{1/4}} \mu_0 .$$

Deep-inelastic data indicate μ_0 is about 500 MeV. Within the bag model, Johnson and Thorn[16] have calculated α' in terms of B and α_s:

$$\alpha'_{\text{bag}} = \frac{1}{8\pi^{3/2}} (\tfrac{3}{2})^{1/2} \frac{1}{\sqrt{\alpha_s B}} .$$

This gives 0.9 GeV^{-2} for $\alpha_s = 2.2$ and $B^{1/4} = 150$ MeV. For $\alpha_s = 1$ the correct slope would correspond to $B^{1/4} = 330$ MeV. So c would have to be 8 to give a completely consistent relationship. We regard this as uncomfortably large, but putting $c = 1$ gives a factor of two wrong for $B^{1/4}$ and perhaps our calculations are just too crude to expect better agreement.

In the foregoing discussion of finite $1/N_c$ effects, we have implicitly assumed that large but finite N_c still yields a confining theory. We know that these effects work against the confining mechanism. For example for $N_c = 1$ we know that all the complicated $1/N_c$ effects sum up to cancel all self-gluon interactions leaving a theory of free quarks and gluons for $SU(N_c)$ or a presumably nonconfining Abelian theory for $U(N_c)$. The $1/N_c$ expansion describes a continuous range of theories from a normal gauge theory ($N_c = 1$) to a confining one ($N_c = \infty$). Our scheme therefore includes the conjecture that there is a critical value of N_c below which confinement is lost. It is a logical possibility that this value is $N_c = \infty$ so that our mechanism would not confine for any finite N_c. Whether or not this is so is of course a complicated dynamical question.

To conclude, we have in this article outlined a new approach to the problem of quark confinement. It is strongly motivated by intuition gained from dual string models and in particular the fishnet idea of Nielsen and Olesen and Sakita and Virasoro[17] as elaborated by 't Hooft and more recently by Thorn, and Brower, Giles and Thorn.[14] It is amusing that the scheme yields a relationship between the fundamental hadronic scales, to wit the Regge slope α', and the scale in deep-inelastic lepton scattering μ_0. The calculations we have outlined form a crude bridge between these two very different kinematic domains. There is a real hope that our formulation can make meaningful contact with the whole range of phenomena which come under the name of strong-interaction physics.

Note added in proof. We have recently recognized that the effects of the closed-string condensate described in Sec. V are in fact present in the sum over all possible P^+ distributions for the gluons in the $N_c \to \infty$ graphs. Thus the tachyon instability might well be removed if the $N_c \to \infty$ limit is not treated approximately. In an approximate treatment, such as given in Sec. IV where the P^+ of each gluon is assumed to be constant, the instability will be present unless the condensate is put in explicitly, for example, along the lines described in the latter part of Sec. V. We shall discuss these matters in more detail in a future communication.

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APPENDIX

In this appendix we calculate the repulsive contact interaction between closed strings due to the four-gluon term in P^-, viz.,

$$P^-_{4G} = \frac{-g^2}{2} \int d\vec{x}\, dx^- \, \mathrm{Tr}[A_i A_j A_i A_j - A_j A_i A_i A_j]\,. \tag{A1}$$

The bare closed string (i.e., $N_c \to \infty$) can be represented by a state of the form

$$|1 \text{ string}\rangle = \sum_l c_l \, \mathrm{Tr}[a_1^+ a_2^+ \cdots a_l^+]|0\rangle\,,$$

where we have suppressed all labels, $\vec{x}$, P^+, i. A state describing two bare closed strings is therefore of the form

$$|2 \text{ strings}\rangle = \sum_{l,m} c_l c'_m \, \mathrm{Tr}\,[a_1^+ a_2^+ \cdots a_l^+] \times \mathrm{Tr}[a_{1'}^+ a_{2'}^+ \cdots a_{m'}^+]|0\rangle\,. \tag{A2}$$

So consider the application of (A1) to a typical term in (A2):

$$\begin{aligned} P^-_{4G}\, \mathrm{Tr}[a_1^+ \cdots a_l^+]\, \mathrm{Tr}[a_{1'}^+ \cdots a_{m'}^+]|0\rangle = \frac{g^2}{2}\frac{8}{N_c} \int d\vec{x}\, dx^- \frac{1}{2\pi(4P_1^+P_1^{+\prime})^{1/2}} e^{-ix^-(P_1^+ + P_1^{+\prime})}\delta(\vec{x}_1 - \vec{x}_{1'}) \\ \times \{\mathrm{Tr}[A_{i_1} a_2^+ \cdots a_l^+]\, \mathrm{Tr}[A_{i_{1'}} a_{2'}^+ \cdots a_{M'}^+] \\ - \mathrm{Tr}[A_{i_{1'}} a_2^+ \cdots a_l^+]\, \mathrm{Tr}[A_{i_1} a_{2'}^+ \cdots a_{M'}^+]\}|0\rangle \\ + \text{cyclic permutations} + \text{noncontact terms.} \end{aligned} \tag{A3}$$

In (A3) the noncontact terms include contractions which do not tie together the two traces and those which do not reproduce a state of identical structure to the original state. We are picking out the part of P^-_{4G} which leaves the internal structure of the two closed strings unaltered. The factor of $1/N_c$ comes from projecting back onto the color-singlet configurations of closed strings. If each of the closed strings originally have spin 0, we must consider only the spin-zero projection of the two factors in (A3). Then the second term will yield minus $\frac{1}{2}$ the first and we obtain

$$\begin{aligned} P^-_{4G}\, \mathrm{Tr}[a_1^\dagger \cdots a_l^\dagger]\, \mathrm{Tr}[a_{1'}^\dagger \cdots a_{m'}^\dagger]|0\rangle = \frac{g^2}{2}\left(\frac{4}{N_c}\right)\delta(\vec{x}_1 - \vec{x}_1') \int_0^{P_1^+ + P_1^{+\prime}} \frac{dQ^+}{2\pi[16P_1^+P_1^{+\prime}Q^+(P_1^+ + P_1^{+\prime} - Q^+)]^{1/2}} \\ \times \mathrm{Tr}[a_1^\dagger(\vec{x}_1, Q^+)a_2^\dagger \cdots a_l^\dagger] \\ \times \mathrm{Tr}[a_{1'}^\dagger(\vec{x}_1', P_1^{+\prime} + P_1^+ - Q^+)a_{2'}^\dagger \cdots a_{M'}^\dagger]|0\rangle \\ + \text{cyclic permutations} + \text{noncontact terms.} \end{aligned}$$

For comparison, the repulsive contact interaction between two scalar quanta in ϕ^4 theory is

$$\begin{aligned} \frac{\lambda}{4!}\int d\vec{x}\, dx^- \phi^4 a^\dagger(\vec{x}_1, P_1^+)a^\dagger(\vec{x}_2, P_2^+)|0\rangle = \tfrac{1}{2}\lambda\delta(\vec{x}_1 - \vec{x}_2) \int_0^{P_1^+ + P_2^+} \frac{dQ^+}{2\pi[16P_1^+P_2^+Q^+(P_1^+ + P_2^+ - Q^+)]^{1/2}} \\ \times a^\dagger(\vec{x}_1, Q^+)a^\dagger(\vec{x}_2, P_1^+ + P_2^+ - Q^+)|0\rangle\,. \end{aligned}$$

Our crude analog [Eq. (5.1)] ignores the composite nature of the closed strings, and it is not clear what effective λ we should choose. At our present level of crudeness we make the blind identification

$$\lambda_{\text{eff}} = \frac{4g^2}{N_c} = 2(4\pi\alpha_s)\frac{1}{N_c} \tag{A4}$$

[note: with our normalization (2.5), $\alpha_s = g^2/2\pi$].

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